Quiz 2a

Answer all questions and make your presentation clear and clean. Each question carries 10 marks.

1. Let D be the region in the first quadrant bounded by $xy = 1, xy = 16, y = 4x, y = 9x$. Evaluate

$$\iint_D \left(\frac{y}{x}\right)^{1/3} dA .$$

$u = xy \in [1, 16]$

$v = y/x \in [4, 9]$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = \frac{2y}{x} = 2v, \therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}.$$

$$\text{Integral} = \int_1^{16}\int_4^9 v^{1/3}\frac{1}{2v}\,dv\,du = \int_1^{16} \frac{1}{2}\frac{1}{1-\frac{2}{3}} v^{\frac{1}{3}}\Big|_4^9 du = \frac{45}{2}\left(9^{\frac{1}{3}} - 4^{\frac{1}{3}}\right) \#$$

2. Evaluate the line integral

$$\int_C xy\, ds$$

where C is the line segment between $(0, 0, 1)$ and $(-1, 2, 3)$. $t \in [0,1]$

$\vec{r}(t) = (0,0,1) + t[(-1,2,3) - (0,0,1)] = (-t, 2t, 2t+1)$ describe the line,

$\vec{r}'(t) = (-1, 2, 2) \quad |\vec{r}'(t)| = \sqrt{9} = 3$

$$\therefore \text{line integral} = \int_0^1 (-t)(2t)\, 3\, dt = -2 \#$$

3. Find the circulation and flux of the vector field

$$\mathbf{F} = x\mathbf{i} + xy\mathbf{j}$$

around and across the unit circle $x^2 + y^2 = 1$ in anticlockwise way.

$\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [0, 2\pi]$ describe the circle.

$\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

$$\therefore \text{circulat} = \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (\cos t\,\hat{i} + \cos t \sin t\,\hat{j})\cdot(-\sin t\,\hat{i} + \cos t\,\hat{j})\, dt$$

$$= \int_0^{2\pi} (-\cos t \sin t + \cos^2 t \sin t)\, dt = 0 .$$

$$\text{flux} = \oint_C N\,dx + M\,dy = \int_0^{2\pi} [\cos t \sin t(-\sin t) + \cos t \cos t]\, dt$$

$$= \pi .$$

Quiz 2b

Answer all questions and make your presentation clear and clean. Each question carries 10 marks.

1. Evaluate the integral

$$\int_0^3 \int_{y=x^2+1}^{y=x^2+4} \sqrt{y-x^2}\, dydx \ ,$$

making use of the change of variables formula.

Let $u = x$, $v = y - x^2 \in [1, 4]$. $\quad \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ -2x & 1 \end{vmatrix} = 1$

Integral $= \int_0^3 \int_1^4 \sqrt{v} \cdot 1 \cdot dv\, du$

$= 14$.

2. Find the mass of the wire described by $x = 3\cos t, y = 3\sin t, z = 4t, t \in [0, \pi]$, with density $\delta = 2z$.

$\vec{r}(t) = (3\cos t, 3\sin t, 4t)$

$\vec{r}'(t) = (-3\sin t, 3\cos t, 4)$

$|\vec{r}'(t)| = \sqrt{9+16} = \sqrt{25} = 5$

mass $= \int_0^{\pi} 2 \times 4t \times 5\, dt = 20\pi$.

3. Find the work done of the vector field $-y\mathbf{i} + x\mathbf{j}$ in moving a particle clockwise once around the unit circle in the xy-plane.

$\vec{r}(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$ anticlockwise.

$\vec{r}'(t) = (-\sin t, \cos t)$

Work $= \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = 2\pi$

For clockwise direction, work done is -2π #